ADAM Research

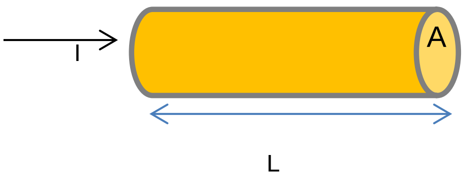
White Paper No. 10
The Adiabatic Wire: "Onderdonk"

Questions: How are temperature, time and electric current correlated?

When does a copper wire melt?

What is the background of the “Onderdonk Equation”?

What are the limitations and how to extrapolate to other environments?



Letters and Units.

I	(DC) current	Ampere
t , Δt	Time , time interval	sec
h	Heat exchange coefficient	W/m^2K
T	Wire temperature (bulk)	°C
T_{ini}	Initial wire temperature	°C
Θ	Temperature rise $T – T_{ref}$	Kelvin
s	Circumference of wire	m
r	Radius of wire	m
L	Length of wire	m
V	Volume of wire	m^3
M	Mass of wire	kg
R	Electric resistance of the wire	Ohm
c_p	Specific heat capacity (copper)	J/(kg*K)
ρ	Mass density of wire material (copper)	kg/m^3
ρ_{ref} , ρ_{20}	specific electric resistivity (copper)	$Ohm*m^2/m$
α_{ref} , α_{20}	slope in copper resistivity formula	1/K
σ	Stefan-Boltzmann constant $=5.67\cdot10^{-8}$	W/m^2K^4
ε	Surface radiative emissivity	-

1. Preface

The time when a wire in a fuse or a trace on a printed board melts is said to be given by “Onderdonk’s equation”. To say it before: this is not perfectly true.

The equation which we will re-derive below will give an answer by numbers, but the basic assumption is strict so that it won’t apply to every perceivable application. It even cannot predict whether a wire will melt at all! This is because only heating by electricity is taken into account but all cooling and phase change effects are excluded. This is not realistic, if the fusion time is long plus the wire has a cool support or the wire is a trace on a printed board. Moreover the wire will melt first in the midpoint, whilst the equation does not care about the position on the wire. Onderdonk’s Eq. can help to give some estimate, but always keep the cooling question in mind.

2. Assumptions

- We consider the case of a *very short pulse of current*. Then we may assume that Joule heat which is created in the conductor has no time to leave the wire, neither by radiation nor by convection (“*adiabatic approximation”)* (Stauffacher, 1926). Heat transfer has not to be considered in this case, but a posteriori we shall have to check what “*short*” means.
- The material conductivity is infinitely large, so that a bulk treatment is possible.
- We take not into account the time needed to melt the wire completely, i.e. we don’t use the melting enthalpy.
- All material properties shall be independent of temperature, except the electric specific resistivity of copper. This point was not yet considered in our White Paper #8.

3. Basic Equation

The following equation describes the

Gain of thermal energy (l.h.s.) = Joule heating by electric energy (r.h.s.)

$$c_p \cdot M \cdot \Delta T = R \cdot I^2 \cdot \Delta t$$

in a bulk of matter over a time span Δ*t* (starting at *t*=0). Because there is no heat loss over the surface, temperature *T* would rise infinitely with time.

The unit on both sides is W·s=Joule. *M* and *R* depend on the geometry of the wire, c_p is a material value. The electric resistance of a wire is temperature dependent and is typically written as

$$R(T) = \rho_{20}(1 + \alpha_{20} \cdot (T - 20))\frac{L}{A} \quad , \tag{1}$$

linked to a reference temperature of 20 °C. Using the basic relation for volume and mass we rewrite

$$c_p \cdot \rho \cdot L \cdot A \cdot \Delta T = \rho_{20}(1 + \alpha_{20}(T - 20))\frac{L}{A} \cdot I^2 \cdot \Delta t$$

The length of the wire cancels. If we assume the initial temperature of the wire being T_{ini}=20 °C, we can unify ΔT and T-20 to a single temperature rise variable Θ. Because of the permanent link between t and Θ, Θ(t) will be time-dependent and the basic equation is an ordinary differential equation

$$\boxed{c_p \cdot \rho \cdot \frac{\mathrm{d}\Theta}{\mathrm{d}t} = \rho_{20}(1+\alpha_{20}\Theta)\frac{I^2}{A^2}} \quad . \tag{2}$$

I/A is also called current density (A/m²). Although Eq. (1) contains geometric parameters, the problem is treated point-like without solving for dimensions in space.

Our material data of copper at 20 °C will be:

ρ_{20}=0.0175 Ω mm²/m (at 20 °C) or $1.75{\cdot}10^{-8}$ Ω-m, resp., and α_{20}=0.00395 1/K, (at 20 °C) and c_p=385 J/kgK , ρ=8900 kg/m³.

4. Solving the Equation

We solve Eq. (2) step by step. First we collect the parameters in an auxiliary quantity K

$$K = \frac{\rho_{20}}{c_p \cdot \rho}\frac{I^2}{A^2}$$

which leads to

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = K(1+\alpha_{20}\Theta)\,.$$

Then we substitute

$$\vartheta := 1+\alpha_{20}\Theta \;\;\rightarrow\;\; \frac{d\Theta}{dt} = \frac{d\Theta}{d\vartheta}\frac{d\vartheta}{dt} = \frac{1}{\alpha_{20}}\frac{d\vartheta}{dt}$$

and get a simple ODE

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}t} = \alpha_{20}K\vartheta\,. \tag{3}$$

Initial condition: at time t=0 the wire shall be at 20 °C (again):

Initial condition Θ(t=0)=0 -> ϑ(t=0)=1

Keep in mind that 20 °C is used twice: first, in the definition of the resistance (1) and second as initial condition (see also below and the Appendix). The way to solve Eq. (3) is straightforward. We integrate from initial state onward to time t

$$\int_1^{\vartheta}\frac{\mathrm{d}\vartheta'}{\vartheta'} = \int_0^t \alpha_{20}K\mathrm{d}t'$$

which gives

$$\ln(\vartheta) - \ln(1) = \alpha_{20}K(t-0)$$

or (because ln(1)=0)

$$\ln(\vartheta) = \alpha_{20} K t \,. \quad (4)$$

Eq. (4) already is the “Onderdonk-Equation” in short-hand notation. Before we show the original version, we first unfold the auxiliary variables ϑ and K

$$\ln(1+\alpha_{20}\Theta) = \alpha_{20} \frac{\rho_{20}}{c_p \cdot \rho} \frac{I^2}{A^2} \cdot t$$

and exchange rhs and lhs

$$\boxed{\frac{\alpha_{20}\rho_{20}}{c_p \cdot \rho} \frac{I^2}{A^2} \cdot t = \ln(1+\alpha_{20}\Theta)} \;. \quad (5)$$

If we write α_{20}=0.00395 like α_{20}=1/253 then the similarity with “Onderdonk”

Original Onderdonk Equation : $$33\left(\frac{I}{A}\right)^2 S = \log_{10}\left(\frac{t}{274}+1\right) \quad (6)$$

becomes obvious (Eq. (6) taken by cut and paste from Stauffacher,1928) (see historic note at the end of the paper). In Stauffacher’s notation *S* is time in seconds and *t* is the temperature *rise* in copper. *A* is the cross-section given in circular(!)-mils. Whether 1/253 or 1/274 is correct will be discussed later.

The temperature rise $\Theta(t)$ from Eq. (5) as a “usual” function would read as

$$\Theta(t) = \frac{e^{\frac{\alpha_{20}\cdot\rho_{20}}{\rho\cdot c_p}\cdot\left(\frac{I}{A}\right)^2\cdot t} - 1}{\alpha_{20}} \;. \quad (7)$$

We don’t know Onderdonk’s set of material data. To check the coefficient in front of $(I/A)^2$ let us use our (‘modern’) material data and convert Eq. (5) to circular-mils and $\log_{10}$

a) SI Units : $\frac{\alpha_{20} \cdot \rho_{20}}{c_p \cdot \rho}$ = 0.00395 · 1.75·10^{-8} /(385·8900) = 2.0·10^{-17}
b) 1 m² =1.97 10^{9} circ-mils
c) We have to compensate $A^2 \rightarrow$ 1 m^4 = 3.9·10^{18} circular-mils²
d) Introducing to a) → 2.0·10^{-17} · 3.9·10^{18} = 79
e) From ln to $\log_{10}$ we have to divide by ln (10) = 2.30 → 79/2.30 = 34

So, a “modern” version, with *A* in circular-mils, comes out to be

$$34\left(\frac{I}{A}\right)^2 \cdot t = \log_{10}\left(\frac{\Theta}{253}+1\right) \;. \quad (8)$$

The coefficient in Eq. (8) is almost the one from Eq. (6). The biggest difference is 1/274 compared with 1/253 in the log-term. Thanks to a precise analysis by D. Brooks we discovered that this has to do with the choice of the reference (initial) temperature (see Appendix): Onderdonk’s reference was 40 °C, our derivation used 20 °C as a reference. This changes the value in the denominator of the log term!

For a general initial temperature the Onderdonk Equation in circular-mils must be written as

$$34\left(\frac{I}{A}\right)^2 \cdot t = \log_{10}\left(\frac{T-T_{ini}}{233+T_{ini}}+1\right) \quad (A \text{ in circ-mils}) \qquad (9a)$$

or with cross-section A in mm² (including conversion from ln to $\log_{10}$)

$$\boxed{8.9\cdot10^{-6}\left(\frac{I}{A}\right)^2 \cdot t = \log_{10}\left(\frac{T-T_{ini}}{233+T_{ini}}+1\right)} \quad (A \text{ in mm}^2). \qquad (9b)$$

The time to reach the onset of copper melting t_{melt} can then be derived from the relative melting point of Θ_{melt}=1085 °C - T_{ini} and Eq. (9b) from

$$\boxed{t_{melt} = 1.15\cdot10^5 \log_{10}\left(\frac{1085-T_{ini}}{233+T_{ini}}+1\right)\left(\frac{I}{A}\right)^{-2} = C\cdot\left(\frac{I}{A}\right)^{-2}} \quad (A \text{ in mm}^2) \quad (10)$$

with the following coefficients

Table 1: Coefficient in Eq. (10) for various initial wire temperatures

Initial temperature (°C)	Coefficient *C* (*A* in mm²)
20	$8.2\cdot10^4$
80	$7.2\cdot10^4$
115	$6.5\cdot10^4$

5. Limitations

It must be repeated again that Eqs. (5,6,7,8,9,10) only hold if there is no heat loss at all out of the wire. However, as the wire heats up there must be a heat loss by radiation, by conduction and by convection. We try to estimate a time up to then this requirement is almost fulfilled.

a) Radiation: Because radiation has the least "inertia" we add a negative term to our energy balance equation which removes heat according to the Stefan-Boltzmann radiation law over the wires' surface

$$c_p \cdot \rho \cdot L \cdot A \cdot \frac{\mathrm{d}T}{\mathrm{d}t} = \rho_{el}\,(1+\alpha(T-20))\,\frac{L}{A}I^2 - \varepsilon\sigma Ls((T+273)^4-(20+273)^4)$$

We added also new variables: the emissivity of the surface ε, the Stefan-Boltzmann constant σ and the surface area $L\cdot s$. Note that the radiation law works on the Kelvin scale only. Fortunately we do not need to consider the heating part for purpose, but the 4th power makes it impossible to write an exact equation in terms of Θ(t). The workaround is to linearize the loss term like $-\varepsilon\sigma f_{rad} Ls\cdot\Theta$, with an approximate radiation factor $f_{rad.}$ Having done this, the remaining terms are

$$c_p \cdot \rho \cdot \frac{\mathrm{d}\Theta}{\mathrm{d}t} "\cong" -\varepsilon\sigma \frac{s}{A} f_{rad}\Theta$$

s/A for a circular wire is $2\pi r/r^2\pi = 2/r$. The solution of this “equation” is an exponentially decaying curve in time ($e^{-t/\tau}$) with time constant τ_{rad}

$$\tau_{rad} = \frac{c_p \cdot \rho}{\varepsilon\sigma f_{rad}} \frac{r}{2}$$

What to choose for f_{rad}, because radiation so much temperature dependent? Because our aim is not accuracy we take the simplest expression $f_{rad}=4\cdot T^3_m$, with $T_m=(T_{melt}+293)/2$ (in Kelvin).

As an arbitrary example we consider a copper wire of radius r=0.1 mm. The melting temperature of copper is T_{melt}=1085 °C= 1355 K. which gives $f_{rad}=2\cdot10^9$. The emissivity of a polished wire is almost 0, that of an oxidized wire about 0.9; so we chose ε=0.5.

Then $\tau_{rad} \approx 385\cdot3900\cdot0.0001 /(0.5\cdot5.67\cdot10^{-8}\cdot2\cdot10^9.\ 2) = 150/110 \approx 1$ sec

b) *Convection.* Without derivation we write the convective time scale as

$$\tau_{conv} = \frac{c_p \cdot \rho}{h} \frac{r}{2} \ ,$$

with h being the convective heat transfer coefficient (see White Paper 1). h is of the order of 5 to 20 W/m²K, For our r=0.1 mm wire it comes out to be around $\tau_{conv} \approx 3$ sec or a bit longer.

c) *Conduction.* In White Paper 8 we derived how far heat can spread out in a time span Δt. This is controlled by the thermal inertia and the thermal conductivity k of the surrounding mediium

$$\Delta t \approx \frac{\rho C}{k} (\Delta x)^2$$

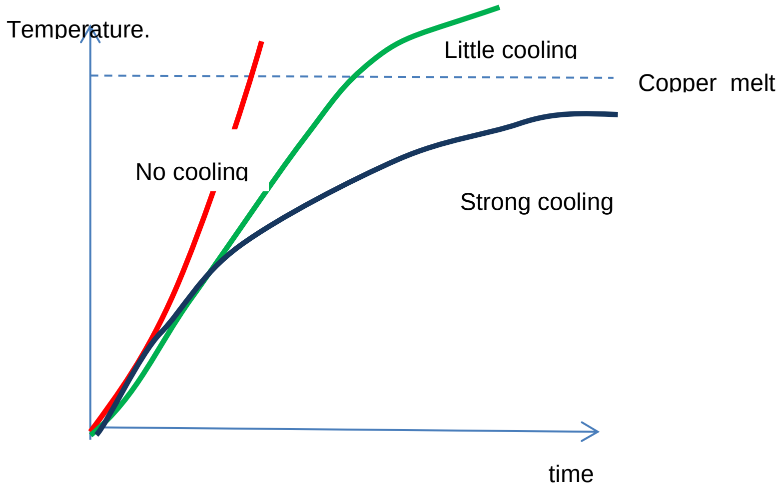
If we consider a wire in *air* and assume that a travelling distance of $2r$ already shows the effect of heat removal, then $\Delta t \approx 1000*1200/0.026 * 0.0002^2 = 2$ seconds.

All three time constants are of the order of *few seconds*. As we know from exponential curves one time constant is already a significant span of time and we see deviations from a linear slope. So it is safe to say:

“Onderdonk is valid until fuse time of much smaller than τ_{rad}”

, which is ≈ 0.1 seconds in our example. For a wire radius of 1 mm it will be around 1 s.

There is even a worse story to be told. If we allow for realistic cooling: the wire may become hot but not hot enough to melt. That depends on geometry and the thermal conditions around the wire, not only on current or wire diameter.



To avoid this delicate tuning, never read the Onderdonk numbers to more than 1 digit and allow for a substantial **safety factor.**

d) Equation (5) was derived under the assumption of constant current (DC). However, in a short-circuit pulse inductive effects are leading to a non-constant signature of $I(t)$. Then K would have to contain a $\int I^2\,dt$ term.

6. Application

Let us plot Eq. (7) for several values of I (Amp) and A (mm²) for an initial temperature of 20 °C. The labels on the vertical axis are denoted as $\Theta(t, I, A)$. We use a log-log representation to compress both time and T axis.

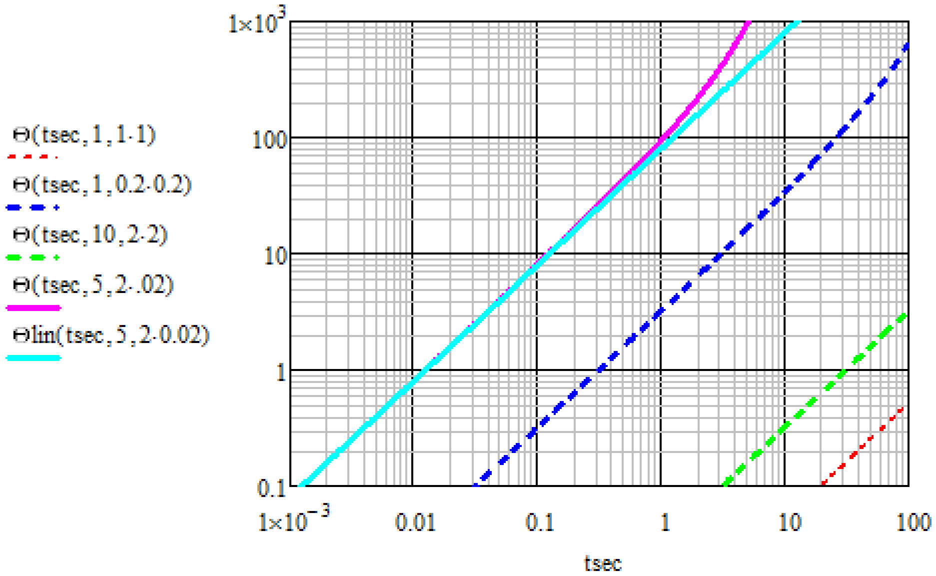


Figure 1: "Onderdonk solutions" of an insulated wire: temperature rise (K) vs. time (s). The pink line represents any (A=0.04 mm²) wire with 5 Amps.

What can we infer?

- Only the leftmost scenario (5 A, 2 mm x 0.2 mm = 0.04 mm² wire) (the pink line) has little chance to fulfill the adiabatic requirement (t_{melt} within few seconds). The higher the current or smaller the cross-section the better.
- Eq. (10) gives t_{melt}=8.2·10^{4}·(5 A / 0.04 mm²)$^{-2}$ = 5.2 seconds (in agreement with the time value for 1000 K of the pink line in Fig. 1).
- The line in light-blue is the solution of Eq. (2) without taking into account the temperature-dependence of the electric resistivity (α_{20}=0). This is a linear function in time and has a slope of 1 in log-log. Comparing with the pink line we see that due to the temperature coefficient α_{20} the melting time is reduced substantially.

Although the software is not dedicated to this type of situation, we can try to perform numerical simulations using the **TRM** electro-thermal simulation software from ADAM Research. The geometry is just a strip made of copper with suppressed cooling to the ambient (h=0 W/m²K). The next figure compares simulation (symbols) and theory (lines). Although, due to numerical reasons, the symbols lag little behind theory, the agreement is good enough.

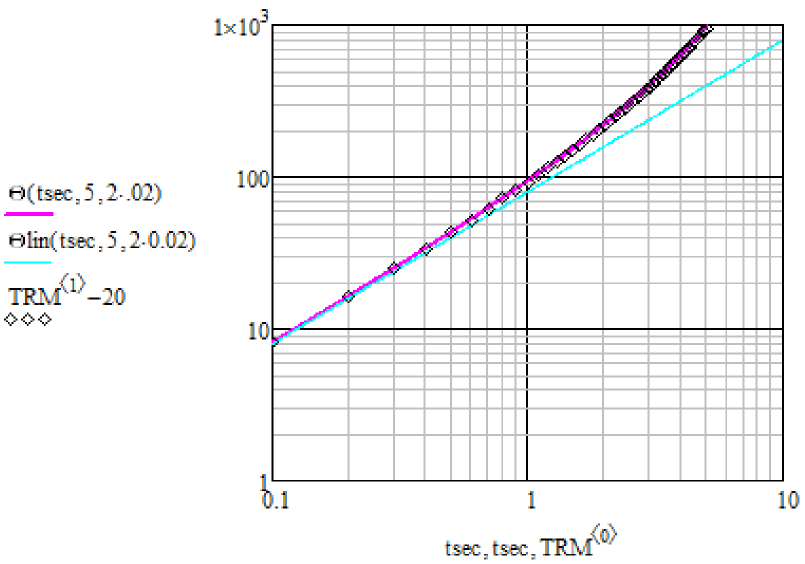


Figure 2: Numerical simulation of a heated and insulated wire with TRM (symbols) vs. analytic solution (pink line= Eq.(7)) for 5 Amps and cross-section 0.04 mm².

7. Historical Note

While working with Douglas Brooks on fusing, the question about the origin and reliability of Onderdonk's equation arouse. Some authors referenced an Onderdonk (1944) paper and a paper Stauffacher&Onderdonk (1928). Searching in libraries, Douglas found the original prints and - surprisingly - none of them had I.M.Oderdonk as an author. It was always said "developed by Onderdonk". See more in Brooks (2015).

8. Summary and Warnings

"Onderdonk's equation" is the transient analytic solution of the temperature rise in a thermally insulated wire which is heated internally by electric DC current.

- Because of this assumption its use should be really restricted to very short current pulses (maybe much shorter than one second. See White Paper No.8). If there is substantial heat flow from the wire to its surroundings (as it is the case for traces on printed boards) a current may be never high enough to melt. A real wire will always melt later than predicted with “Onderdonk”. We only can give the shortest possible time, not the real time needed to reach the melting point.
- The formula contains (somewhat hidden) the initial temperature of the wire. This has an effect on the melting time.
- Last but not least the equation does not predict when a wire really got molten. It predicts only the onset of melting, i.e. the time when the melting temperature is reached (melting enthalpy addressed by Basauskas and Wichman, 2011).

Appendix

Eqs. (6) and (8) have different sums in the log term but the coefficients are very close to each other. This is because different reference/initial temperatures are used and this results in another value of α. The reference temperature (e.g. 20 °C) is used twice: for initial temperature and the resistivity formula reference. Both values must be identical, so that the substitution of variables described in Section 4 works. We want to show now what has to be done, if the initial temperature is changed from one value to the other.

To write a linear function in form of

$$\rho(T) = \rho_{20}(1+\alpha_{20}\cdot(T-20)) \quad \text{(A1)}$$

is legitimate, but not standard. α_{20} is not the slope of the curve! If we perform the multiplication we get

$$\rho(T) = \rho_{20} + \rho_{20}\alpha_{20}\cdot(T-20) \quad \text{(A2)}$$

and see that the true slope is $\rho_{20}\alpha_{20}$ because of $\frac{\rho(T)-\rho_{20}}{(T-20)} = \rho_{20}\alpha_{20}$.

- The coefficient in front of I^2/A^2 is $\rho_{20}\cdot\alpha_{20}$ and is identical in value for any other temperature T_{ref}.
- The coefficient in the $\log_{10}$-term is $1/\alpha_{ref}$ or $1/\alpha_{ini}$, resp. at initial =r reference temperature. Can this variable written in some universal way for any temperature T_{ref}? Brooks (Adam and Brooks, 2015) discovered that from $\alpha_{ref}*\rho_{ref} = \alpha_{20}*\rho_{20}$ the following relations

  $1/\alpha_{ref} = \rho_{ref}/(\alpha_{20}*\rho_{20}) = (\rho_{20} + \rho_{20}*\alpha_{20}*(T_{ref}-20))/(\alpha_{20}*\rho_{20})$

  $= \rho_{20}/(\alpha_{20}*\rho_{20}) + (T_{ref} - 20) = 1/\alpha_{20} + (T_{ref}-20)$

  $= (1/0.00395 - 20) + T_{ref} = 253-20 + T_{ref}$

  lead to

  $$1/\alpha_{ref} = 233 + T_{ref} \quad \text{(A3)}$$

  which is the common element in Eqs. (6), (8) and (10) and solves the mystery of various different notations.

Acknowledement.

This White Paper had not got written without having been triggered and improved in discussions with Douglas Brooks (www.ultracad.com), who advised to view this topic from different angles.

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