ADAM Research

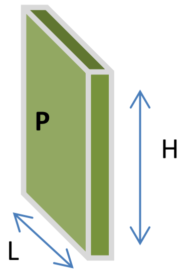
White Paper No. 9

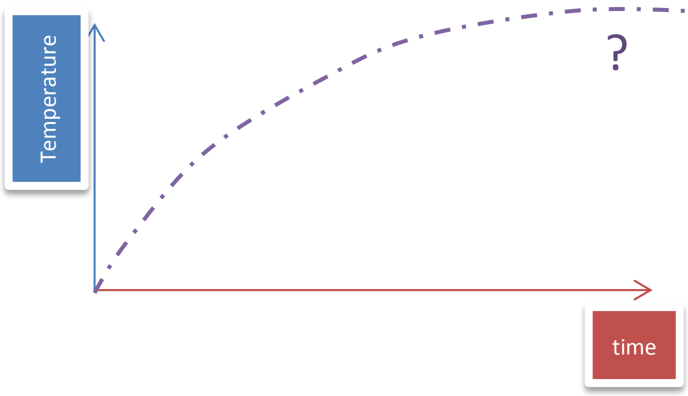
Temperature of a Heated Plate as Function of Time

- Air Cooling and Space Cooling -

Questions: How fast does a plate heat-up?

Is there a difference between cooling in air and cooling by radiation?





Letters and Units.

P	Power dissipation in plate	Watt
t , Δt	Time , time interval	sec
τ	Time constant	s
h	Heat transfer coefficient	W/m^2K
T	Plate temperature (bulk) at time t	°C or K
T_{eq}	Plate temperature (bulk) in equilibrium	°C or K
T_a	Ambient temperature and initial temperature	°C or K
A	Area of plate = 2*L*H	m^2
V	Volume of plate	m^3
M	Mass of plate	kg
c_p	Specific heat capacity (bulk)	J/(kg*K)
ρ	Mass density (bulk)	kg/m^3
σ	Stefan-Boltzmann constant $=5.67 \cdot 10^{-8}$	$W/(m^2K^4)$
ε	Surface radiative emissivity	-
r,x,F	Auxiliary qty.	

1. Introduction

In White Paper No. 1 we discussed the steady-state (final) temperature of a uniformly heated plate. However, when power is "turned on" it will take some time until the equilibrium temperature is attained. This period not only depends on the mass of the plate, but also on the cooling conditions.

How does the heating curve look like? Is there a difference between an operation in air environment or in space? A more elaborate version can be found in [1].

2. Assumptions

- We consider a homogenous, volume heated, perfect thin flat plate without any structure. The heating is permanent and heating power is called *P*.
- The temperature of the plate is represented by a single temperature value *T.*
- Density and specific heat capacity is independent of temperature and time *t*.
- Convection has no inertia. The heat transfer coefficient *h* is constant for all times.
- At *t*=0 the plate is at ambient temperature (initial *T* = ambient *T*). This is onset condition of heating.

3. Basic Equations

The following equation describes (in a one-node-approximation) the energy balance of our plate

Gain of thermal energy = Heating energy – Cooling energy

$$c_p \cdot M \cdot \Delta T = P \cdot \Delta t - P_C \cdot \Delta t \qquad (1)$$

in a time span Δ*t* (starting at *t*=0). Remember: Energy=Power * Time. The unit on both sides is Watt*Second=Joule.

What is the cooling power? Usually, the Newtonian ansatz (see also White Paper No. 1) is used for convection in air: the cooling power is assumed to be proportional to the temperature difference between board and ambient air, proportional to the area of the plate (front + back side) multiplied by a constant of proportionality, called heat transfer coefficient *h*

Air cooling:

$$P_C(T) = h \cdot A \cdot (T - T_a) \qquad (2)$$

It is said that this approach was proposed by Isaac Newton, so we call the result "Newtonian". In natural (buoyancy driven) convection T_a is the surrounding air temperature and moreover the heat transfer coefficient *h* in fact is a function of temperature (so-called Nusselt-Grashof correlations), which we will ignore. In terrestrial situations we typically have a radiative contribution, which we assume to be included in the value of *h*. For moderately heated plates like in electronics, convection and radiation contribute one half to the total value of *h* (around 12 W/m²K) respectively. In pure vacuum environment energy loss is by radiation only and is very much temperature dependent. Typically, the Stefan-Boltzmann law is linearized so that it can be written as a function of T-T_a, thus looking like Eq. (2). This we will not do (so not to reproduce standard text books) but, instead, write the general term

Radiation cooling:

$$P_C(T) = \sigma \cdot \varepsilon \cdot A \cdot (T^4 - T_a^4) \quad . \qquad (3)$$

Of course, σ has not the meaning of a heat transfer coefficient rather it is a combination of nature constants. For Eq. (3) T must be given in Kelvin, in (2) either °C or K work. ε is the emissivity value of the surface (between 0 and 1). We shall assume a perfectly emitting surface coating with $\varepsilon = 1$. T_a now has the meaning of a radiation temperature of walls very far apart.

To follow the course of temperature in time the interval Δ*t* in Eq. (1) is transferred into a differential interval d*t* and so is the temperature increase a differential d*T*.

The equation to be solved reads

$$c_p \cdot M \cdot \frac{\mathrm{d}T}{\mathrm{d}t} = P - P_C \tag{4}$$

In equilibrium the temperature doesn't change anymore and the l.h.s. rise term is zero. The equilibrium temperature can be noted immediately

Convection	*Vacuum*
$0 = P - h \cdot A \cdot (T_{eq} - T_a)$	$0 = P - \sigma \cdot A \cdot (T_{eq}^4 - T_a^4)$
$T_{eq} = T_a + \dfrac{P}{h \cdot A}$	$T_{eq} = \left(T_a^4 + \dfrac{P}{\sigma \cdot A}\right)^{1/4}$
T in Celsius or Kelvin	T in Kelvin

4. Solving the Transient Equation

Without comments we state the solution of Eq. (4). For more details see [1].

Convection	*Vacuum*
$c_p \cdot M \cdot \dfrac{\mathrm{d}T}{\mathrm{d}t} = P - h \cdot A \cdot (T - T_a)$	$c_p \cdot M \cdot \dfrac{\mathrm{d}T}{\mathrm{d}t} = P - \sigma \cdot A \cdot (T^4 - T_a^4)$
$\boxed{\tau := \dfrac{c_p M}{hA}}$	$r := \dfrac{c_p M}{\sigma A}$
$\dfrac{\mathrm{d}T}{T_{eq} - T} = \dfrac{\mathrm{d}t}{\tau}$ (5a)	$\dfrac{\mathrm{d}T}{T_{eq}^4 - T^4} = \dfrac{\mathrm{d}t}{r}$ (5b)

For the convection part we defined a quantity τ, which has the dimension of time: J/(kg*K)*kg/(W/(m²K)*m²) = J/W = Ws/W=s. τ is called *time constant* which will become obvious when we have the solution of (5a). The auxiliary quantity *r* does not have the meaning of a time constant, because its dimension is s/K^4.

By now we have separated the variables *T* and *t* and can integrate both sides independently from initial (cold) state to any time *t*.

$\displaystyle\int_{T_a}^{T} \frac{\mathrm{d}T'}{T_{eq} - T'} = \int_0^t \frac{1}{\tau}\mathrm{d}t'$	$\displaystyle\int_{T_a}^{T} \frac{\mathrm{d}T'}{T_{eq}^4 - T'^4} = \int_0^t \frac{1}{r}\mathrm{d}t'$ (6)

The r.h.s simply is t/τ or t/r, resp. Both l.h.s. can be looked-up in integral tables. To avoid typos and to save space we use a simpler notation. The primitives then are

$\int \frac{\mathrm{d}x}{a-x} = -\ln(a-x)$	$\int \frac{\mathrm{d}x}{a^4-x^4} = \frac{1}{4a^3}\ln\frac{a+x}{a-x} + \frac{1}{2a^3}\operatorname{arctg}\frac{x}{a}$ (7)

The convective (Newtonian) solution can be completed easily ("upper bound – lower bound")

$-\ln(T_{eq}-T) - (-\ln(T_{eq}-T_a)) = \frac{0}{\tau} - \frac{t}{\tau}$	
$\boxed{T(t) = T_{eq} - (T_{eq}-T_a)e^{-t/\tau}}$ (8)	

We check for the boundaries. For t=0: e^0=1 $T(0) = T_{eq} - (T_{eq} - T_a) = T_a$ and, for t=∞: e^{∞}=0 and $T(\infty)=T_{eq}$. The shape of the curve is shown in Fig. 1. Within the duration of 5 time constants τ about 99% of the final temperature is reached.

The preliminary vacuum solution in Eq. (7) looks terrible. If we denote the integral by F(x) the solution formally can be written as

	$\boxed{F(x) = \frac{1}{4T_{eq}^3}\ln\left(\frac{T_{eq}+x}{T_{eq}-x}\right) + \frac{1}{2T_{eq}^3}\operatorname{arctg}\left(\frac{x}{T_{eq}}\right)}$
	$\boxed{F(T) - F(T_a) = t/r}$ (9)

Eq. (9) is an implicit and non-linear equation for $T(t)$ and has to be solved with numerical methods for root finding. Searching for the zero of Eq. (9) can be tricky, because F(x) has very small values and is diverging for $T \rightarrow T_{eq}$. All this should not be discussed here.

5. <u>Examples</u>

5.1. Simple plate in free convection air cooling

We claimed in TRM White Paper No. 1 that a plate of size of a typical PCB has a heat transfer coefficient of h≈10…12 W/m²K, where the natural convective and radiation contributions are added.

According to that, a flat plate (Euro-Format) with a uniform power load and these parameter values

L=0.1 m, H=0.16 m, A=2*L*H=0.032 m², P=10 W, h=11 W/m²K

the plate attains an equilibrium temperature in the lab of

$$T_{eq} = T_a + \frac{P}{h \cdot A} = 20\,°\mathrm{C} + 10\,\mathrm{W} / (11\,\mathrm{W/m^2K} * 0.032\,\mathrm{m^2}) = 20\,°\mathrm{C} + 28\,\mathrm{K} = 48\,°\mathrm{C}.$$

To draw the heating curve Eq. (8), we need to know mass M and specific heat capacity c_P. For simplicity let us assume a 1.6 mm thick plate made of "FR4" (R1566 glass + epoxy) with density 1900 kg/m³ and c_p=1000 J/kgK. Mass M then is 0.05 kg and c_p*M is 48.6. The time constant is

$$\tau := \frac{c_p M}{hA}$$ = 48.6 / (11 * 0.032) = 138 s.

With these numbers we can plot the Newtonian result Eq. (8) in Fig.1.

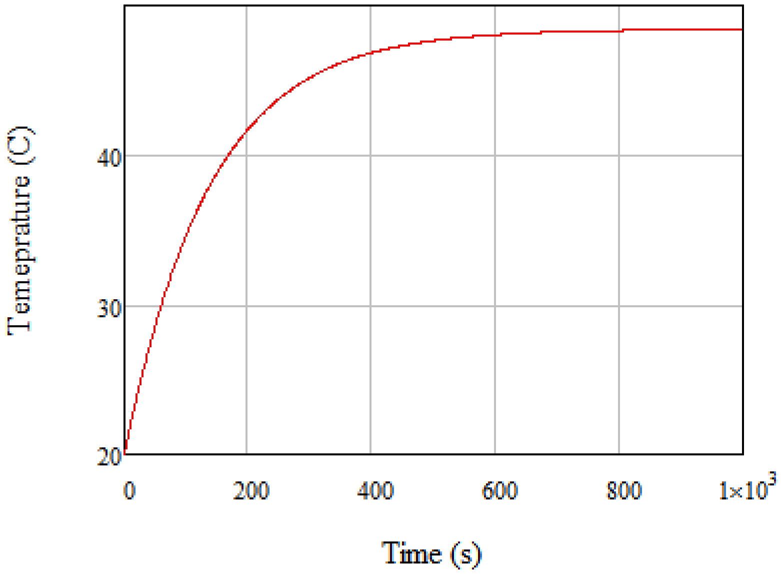


Figure 1: Temperature of the plate in free convection as function of time

This was easy.

5.2. Simple plate in vacuum cooling

We consider the same plate being perfectly black (ε=1) with a uniform power load.

L=0.1 m, H=0.16 m, A=2*L*H=0.032 m², P=10 W

5.1.1 Terrestrial temperature

The plate is placed in a vacuum chamber with wall temperature at 20 °C.

T_a=+20 °C = 293 K

$$T_{eq} = \left(T_a^4 + \frac{P}{\sigma \cdot A}\right)^{1/4}$$ = (293⁴+10/(5.67e-8*2*0.1*0.16)^{0.25} = 336 K = 64 °C

$$h_{eq} = \frac{P}{(T_{eq} - T_a)A}$$ = 10 W / (0.032 m² * 44 K) = 7.1 W/m²K

Fig. 2 contains the graph for Eq. (8) with τ derived from h_{eq} and the numerical solution of Eq. (9) (T rescaled from K to °C). It looks disappointing that both curves are almost identical.

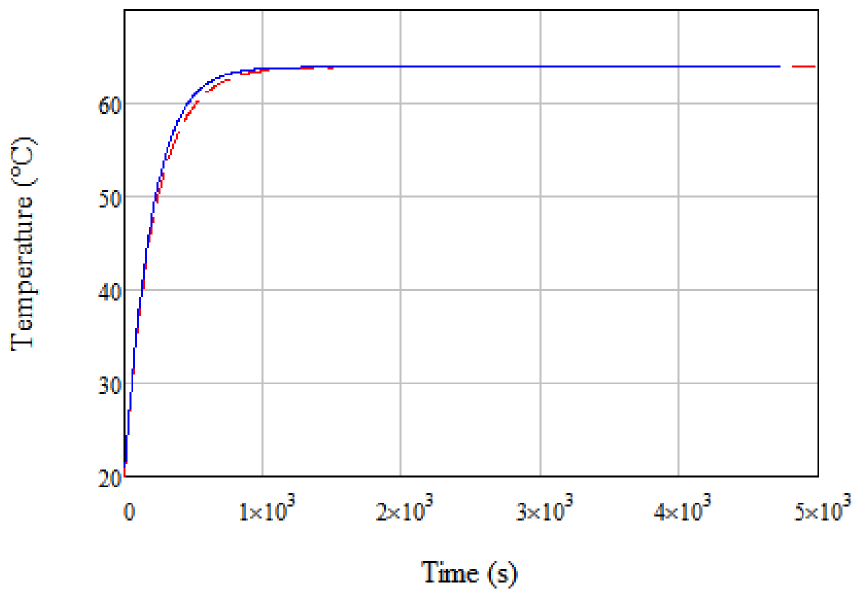


Figure 2: Vacuum solution at terrestrial ambient radiation temperature: Newtonian Eq (8) red and Stefan-Boltzmann Eq. (9) blue

5.1.2 Space temperature

The picture changes immediately when we set the temperature of the ambient radiation field to T_a=-200 °C (regardless whether this a technologically good choice)

$$T_{eq} = \left(T_a^4 + \frac{P}{\sigma \cdot A}\right)^{1/4} = (73^4+10/(5.67e\text{-}8*2*0.1*0.16)^{0.25} = 273 \text{ K} = 0\ °\text{C}$$

$$h_{eq} = \frac{P}{(T_{eq} - T_a)A} = 10 \text{ W} / (0.032 \text{ m}^2 * 200 \text{ K}) = 1.6 \text{ W/m}^2\text{K}$$

The low value of *h* is due to the strong *T*-dependence of radiation: the efficiency of radiative cooling falls rapidly with falling temperature! That is why 10 W lead here to a heating by +200 K above ambient whereas we get only +30 K at room temperature. If we take h_{eq} as input for τ in Eq. (8) the two solutions now are far apart. Even if we draw the red curve with a somewhat different value for τ the curves never match because the initial slope is different.

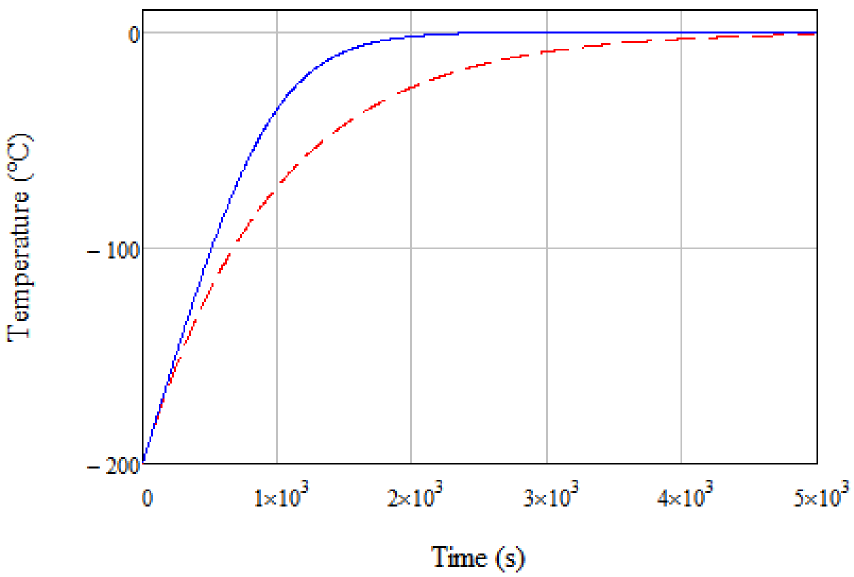


Figure 3: Vacuum solution at space ambient radiation temperature -200°C: Eq (8) red and Eq. (9) blue.

6. Numerical Solution with TRM

Why is it necessary to have an analytical solution? One aspect is to cross-check the results of a numerical simulation. The other one is to get a feeling for sensitivity of parameters and shapes of curves.

6.1 Flat plate

Fig. 4 shows the result of TRM with a very coarse time grid. At the beginning of each time step the the cooling power is recalculated from the surface temperature of the previous one.

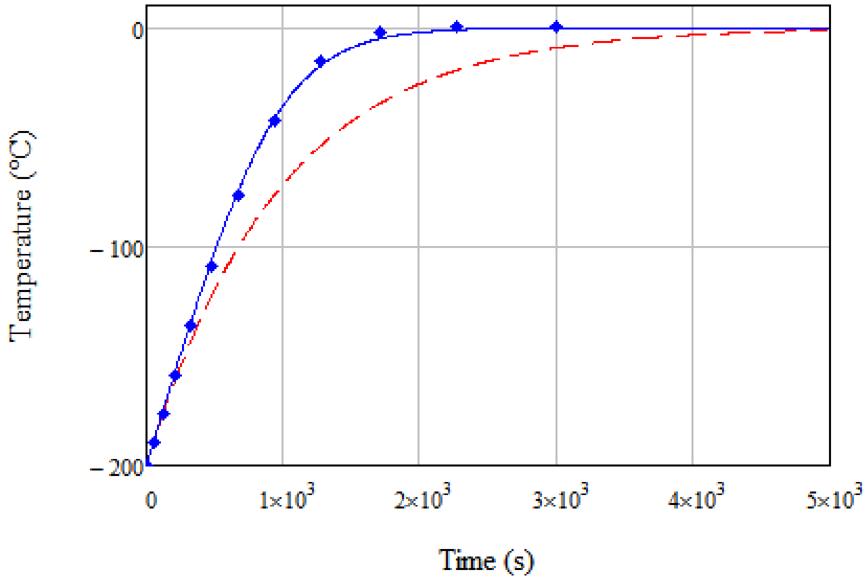
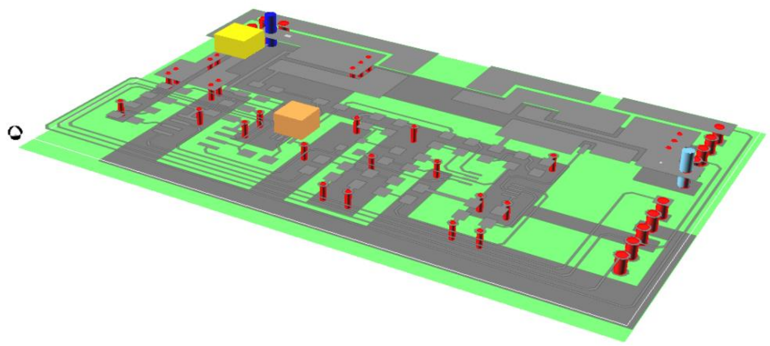


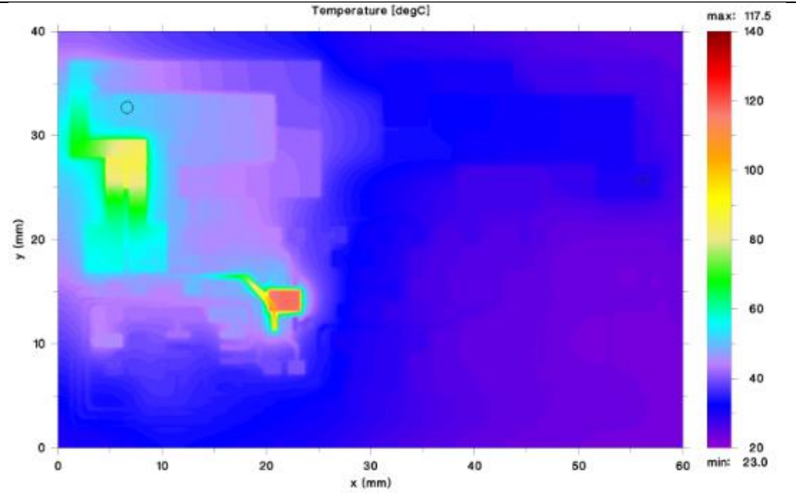
Figure 4: Comparison of analytical result Eq. (9) and TRM simulation values (diamonds).

The simulated temperature is somewhat lagging behind the analytic curve. We don't bother about it because there are other unknowns: what are substrate thermal properties at low temperatures and how do they change with T?

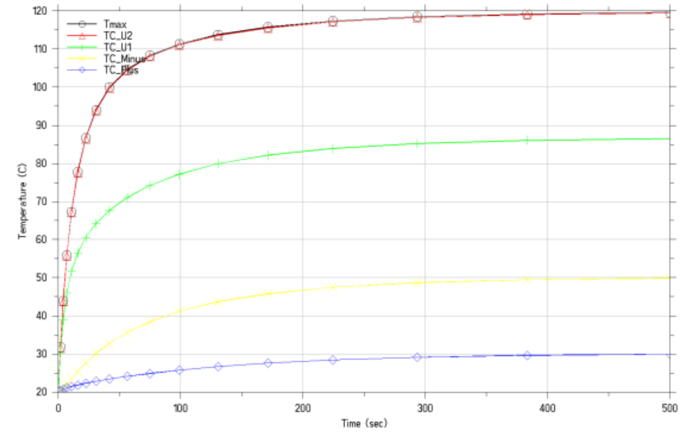
6.2 Printed board with components

Finally, we test the TRM code using the training board "MCP1630" from the TRM User Manual. We compare both ambient assumptions.

	3D view of a 2-layer printed board model "MCP1630" with 2 surface mounted components U1 (yellow) and U2 (orange). Grey contours are layout data. Plated drilled holes in red cylinders.

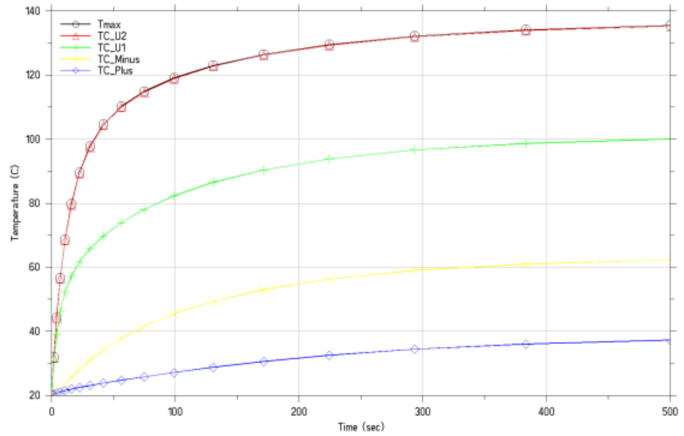


Steady state result in layer 1 with settings from the User Manual (h=11 W/m²K). For the hot component U2 we use 0.2 W.

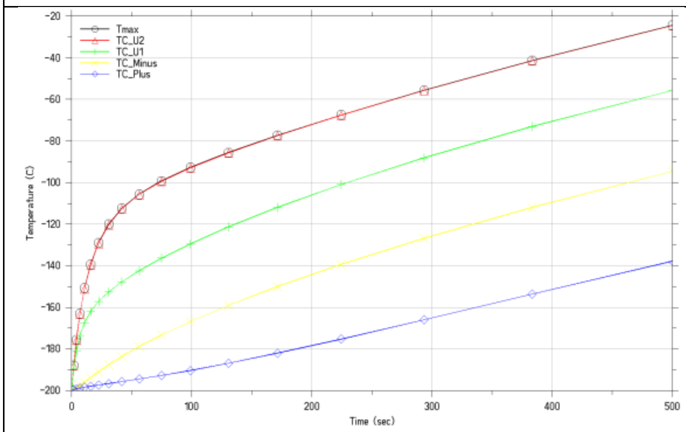


Newtonian Heating curves with h=11 in <u>air</u> at T_a=20 °C .

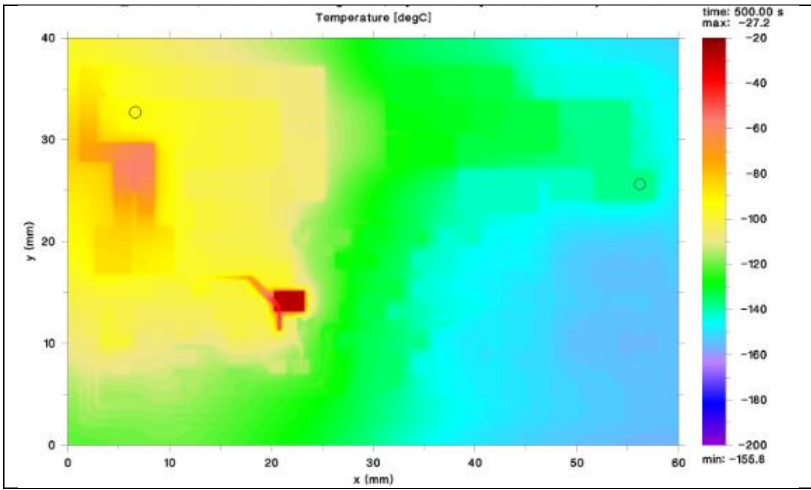
Red/black curve is component U2, green curve is U1.



Stefan-Boltzmann heating curve in <u>vacuum</u> at T_a=20 °C.



Stefan-Boltzmann heating curve in <u>vacuum</u> at T_a=-200 °C, but material data as if 20 °C.



Layer 1 in vacuum at time =500 s

References.

Adam, J., "Time-dependent Heating of Plates – in Air and Vacuum "
http://dx.doi.org/10.13140/RG.2.1.4314.8566

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